Total number of printed pages–24

3 (Sem–5/CBCS) MAT HE 1/HE 2/HE 3

2021

(Held in 2022)

MATHEMATICS

(Honours Elective)

Answer the Questions from any one Option.

OPTION–A

Paper : MAT–HE–5016

(Number Theory)

DSE (H)–1

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

PART–A

1. Choose the correct option : $1\times10=10$

 (i) Two integers *a* and *b* are coprime if there exists some integers *x*, *y* such that

 (a) $ax+by=1$

 (b) $ax-by=1$

 (c) $(ax+by)^n=1$

 (d) None of the above

 (ii) Let $d=\gcd(a,b), n\in\mathbb{N}$. If $d\mid c$ and (x_0,y_0) is a solution of linear Diophantine equation $ax+by=c$, then all integral solutions are given by

 (a) $(x,y)=\left(x_0+\frac{bn}{d}, y_0-\frac{an}{d}\right)$

 (b) $(x,y)=\left(x_0-\frac{bn}{d}, y_0+\frac{an}{d}\right)$

 (c) $(x,y)=\left(x_0+\frac{an}{d}, y_0-\frac{bn}{d}\right)$

 (d) $(x,y)=\left(x_0-\frac{an}{d}, y_0+\frac{bn}{d}\right)$

 (iii) A reduced residue system modulo *m* is a set of integers r_i such that

 (a) $[r_i, m]=1$

 (b) $(r_i, m)=1$

 (c) $(r_i, m)\neq1$

 (d) None of the above

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(iv) Suppose that m_j are pairwise relatively prime and a_j are arbitrary integers $(j = 1, 2, \ldots k)$ then there exist solution x to the simultaneous congruence $x \equiv a_j \left(mod\ m_j\right)$, such that x are

(a) congruent modulo $M = m_1 . m_2 . m_3 \ldots m_k$

(b) congruent modulo $M = \sum_{j=1}^{k} m_j$

(c) congruent modulo m_i

(d) Both (a) and (b)

(v) The product of four consecutive positive integers is divisible by

(a) 20

(b) 22

(c) 24

(d) 26

(vi) Euler's ϕ-function of a prime number p, i.e., $\phi(p)$ is

(a) p

(b) $p-1$

(c) $\frac{p}{2}-1$

(d) None of the above

(vii) For which value of m, CRS $(mod\ m)$ = RRS $(mod\ m)$?

(a) If m is a prime

(b) If m is a composite

(c) If $m < 10$

(d) None of the above

(viii) If $ca \equiv cb (mod\, m)$, then

(a) $a \equiv b \left(mod \frac{m}{(c, m)} \right)$

(b) $a \equiv b (mod\, m)$

(c) $a \equiv b (mod\, m. (c, m))$

(d) None of the above

(ix) The unit place digit of 2^{73} is

(a) 4

(b) 6

(c) 8

(d) 2

(x) The highest power of 7 that divides 50! is

(a) 7

(b) 8

(c) 10

(d) 5

2. Answer the following questions : $2\times5=10$

(a) If p is a prime, then prove that $\phi(p!)=(p-1)\phi((p-1)!)$ 2

(b) Find all prime number p such that p^2+2 is also a prime. 2

(c) For $n=p^k$, p is a prime, prove that

$$n=\sum_{d|n}\phi(d)$$

where $\sum_{d|n}$ denotes the sum over all positive divisors of n. 2

(d) Find the number of zeros at the end of the product of first 100 natural numbers. 2

(e) Find $\sigma(12)$. 2

3. Answer ***any four*** questions : $5\times4=20$

(a) If ϕ is Euler's phi function, then find $\phi(\phi(1001))$. 5

Contd.

(b) Find the remainder, when 30^{40} is divided by 17. 5

(c) State and prove Chinese Remainder Theorem. 5

(d) If p_n is the nth prime number, then prove that

$$p_n<2^{2^{n-1}}$$ 5

(e) If $n=p_1^{k_1}\,p_2^{k_2}\,p_3^{k_3}\ldots p_r^{k_r}$ is the prime factorization of $n>1$, then prove that

(i) $\tau(n)=(k_1+1)(k_2+1)(k_3+1)\ldots(k_n+1)$

(ii) $\sigma(n)=\dfrac{p_1^{k_1+1}-1}{p_1-1}\times\dfrac{p_2^{k_2+1}-1}{p_2-1}\times\ldots\times\dfrac{p_r^{k_r+1}-1}{p_r-1}$

 $2\frac{1}{2}+2\frac{1}{2}=5$

(f) Define Mobius function. Also show that

$$\mu(m\cdot n)=\mu(m)\cdot\mu(n)$$

Hence find $\mu(6)$. $1+3+1=5$

PART–B

*Answer **any four** questions :* $10\times4=40$

4. (a) If $d=(a,n)$, prove that the linear congruence $ax\equiv b\,(mod\;n)$ has a solution if and only if $d\,|\,b$. 5

(b) (i) When a number n is divided by 3 it leaves remainder 2. Find the remainder when $3n+6$ is divided by 3. 2

(ii) Prove that $5n+3$ and $7n+4$ are coprime to each other for any natural number n. 3

5. (a) If p is a prime, then prove that

$$(p-1)! \equiv -1 (mod\ p)$$ 5

(b) Using property of congruence show that 41 divides $2^{20}-1$. 5

6. (a) Prove that every positive integer $(n>1)$ can be expressed uniquely as a product of primes. 5

(b) Determine all solutions in the integers of the Diophantine equation

$$172x+20y=1000$$ 5

7. (a) If n be any positive integer and can be expressed as $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, then prove that $\phi(n) = n \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)$. 5

(b) If m and n are any two integers such that $(m, n) = 1$, prove that

$$\phi(m.n) = \phi(m). \phi(n).$$ 5

8. (a) For each positive integer $n \geq 1$, show that

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n=1 \\ 0, & \text{if } n>1 \end{cases}$$ 5

(b) If k denotes the number of distinct prime factors of positive integer n, then prove that

$$\sum_{d|n} |\mu(d)| = 2^k$$ 5

9. (a) Show that $\sum_{d|n} \mu(d)\, \tau(d) = (-1)^k$

where k denotes the number of distinct prime factors of positive integers n. 5

(b) Prove that

(i) $\tau(n)$ is an odd integer iff n is a perfect square. 3

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(ii) For any integer $n \geq 3$, show that

$$\sum_{k=1}^{n} \mu(k!) = 1.$$ 2

10. (a) Let p be an odd prime. Show that the congruence $x^2 \equiv -1 (mod\ p)$ has a solution if and only if $p \equiv 1 (mod\ 4)$.

(b) If $n \geq 1$ and $gcd(a, n) = 1$, then prove that $a^{\phi(n)} \equiv 1 (mod\ n)$. 5

11. (a) If n is a positive integer and p is a prime, then prove that the exponent of the highest power of p that divides $n!$ is $\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$. 5

(b) Solve $3[x] = x + 2\{x\}$ where $[x]$ denotes greatest integer $\leq x$ and $\{x\}$ denotes the fractional part of x. 5